JOURNAL OF COMPUTATIONAL PHYSICS **63**, 236–239 (1986)

NOTE

A Stable Highly Accurate ADI Method for Hyperbolic Heat Conduction Equation

The classical Fourier theory of heat conduction together with the energy conservation principle leads to the well-known parabolic equation

$$\frac{\partial T}{\partial \theta} = \alpha \nabla^2 T. \tag{1}$$

This equation predicts an infinite speed of propagation of thermal disturbances and hence from the physical aspect is not acceptable. The hyperbolic equation

$$\tau_R \frac{\partial^2 T}{\partial \theta^2} + \frac{\partial T}{\partial \theta} = \alpha \nabla^2 T \tag{2}$$

developed by Cattaneo and Vernotte [1–4], following Maxwell's intuition, has no such drawback and therefore is preferable from the physical point of view. In fact it allows for a finite value of the speed of propagation $(\alpha/\tau_R)^{1/2}$.

Although the effect of a finite speed of propagation is negligible in most practical conventional situations there is a steadily growing interest in Eq. (2). This is due to many special applications where finite wave speed theory can become important [5–10].

The object of this note is to present a stable highly accurate ADI method suitable for numerical solution of the hyperbolic heat conduction Eq. (2), in two space dimensions.

Introducing the following quantities:

$$x = \xi/L, \qquad y = \zeta/L, \qquad T' = T - T_0, \qquad t = \alpha\theta/L^2, \qquad \tau = \tau_R \alpha/L^2,$$

Eq. (2) takes the form

$$\tau \frac{\partial^2 T'}{\partial t^2} + \frac{\partial T'}{\partial t} = \frac{\partial^2 T'}{\partial x^2} + \frac{\partial^2 T'}{\partial y^2}. \tag{3}$$

Since the practical limitation on the spatial accuracy results from the "bandwith" of the system of equations which must be solved after discretization of (3), we restrict our attention to differences that produce tridiagonal systems.

0021-9991/86 $3.00

A simple difference form of (3) based on rational approximation (see [11, 12]) is

$$\tau\left(\frac{\delta_t^2}{1+m\delta_t^2}\right)T_{ij}^n+k\left(\frac{\mu_t\delta_t}{1+m\delta_t^2}\right)T_{ij}^n=p^2\left(\frac{\delta_x^2}{1+s\delta_x^2}+\frac{\delta_y^2}{1+s\delta_y^2}\right)T_{ij}^n. \tag{4}$$

The approximate value of T' at the mesh point ih, jh, nk is denoted by T_{ij}^n and the classical operators: central difference δ and averaging μ are introduced for simplicity. The order of accuracy of difference scheme (4) is $O(h^2+k^2)$ and m, s are arbitrary parameters.

Transforming (4) we get

$$\begin{aligned}(\tau\delta_t^2+k\mu_t\delta_t)\,(1+s\delta_x^2+s\delta_y^2+s^2\delta_x^2\delta_y^2)\,T_{ij}^n\\ =p^2(1+m\delta_t^2)\,(\delta_x^2+\delta_y^2+2s\delta_x^2\delta_y^2)\,T_{ij}^n.\end{aligned} \tag{5}$$

Adding the sixth-order terms $m^2p^4/(2\tau+k)\,\delta_x^2\delta_y^2(\mu_t\delta_t)$ to the left-hand side of (5) and transforming it further, Eq. (5) can finally be written in the split form

$$\begin{aligned}&2\tau^*\mu_t\delta_t[1+(s-mp^2/\tau^*)\,\delta_x^2]\,[1+(s-mp^2/\tau^*)\,\delta_y^2]\,T_{ij}^n\\ &\quad=p^2(1-2m\nabla_t)\,(\delta_x^2+\delta_y^2+2s\delta_x^2\delta_y^2)\,T_{ij}^n+2\tau\nabla_t(1+s\delta_x^2+s\delta_y^2+s^2\delta_x^2\delta_y^2)\,T_{ij}^n,\\ &\qquad\qquad \tau^*=\tau+k/2 \qquad\text{and}\qquad \nabla_t T^n=T^n-T^{n-1}.\end{aligned} \tag{6}$$

The added terms do not alter the accuracy but enable the operator on the left-hand side of (6) to be factorized.

The form of (6) more convenient for direct application is

$$\begin{aligned}&\tau^*\,[1+(s-mp^2/\tau^*)\,\delta_x^2]\,[1+(s-mp^2/\tau^*)\,\delta_y^2]\,T_{ij}^{n+1}\\ &\quad=p^2(1-2m)\,(\delta_x^2+\delta_y^2+2s\delta_x^2\delta_y^2)\,T_{ij}^n+2\tau[1+s(\delta_x^2+\delta_y^2)+s^2\delta_x^2\delta_y^2]\,(T_{ij}^n-T_{ij}^{n-1})\\ &\qquad+\tau^*[1+(s+mp^2/\tau^*)\,\delta_x^2]\,[1+(s+mp^2/\tau^*)\,\delta_y^2]\,T_{ij}^{n-1}.\end{aligned} \tag{7}$$

If $\tau=0$, $s=\frac{1}{12}$ and $m=\frac{1}{2}$ or $\frac{1}{3}$, Eq. (7) takes the form of the known higher accuracy split formulas for parabolic equations [13].

To determine the accuracy of the two parameter family of difference schemes (7), the Taylor expansion of the terms on both sides of (7) is carried out about the reference node (ih, jh, nk). Subtracting the resulting terms on the right-hand side from those on the left-hand side, the principal part of the truncation error is found to be

$$h^4p^2\left[\left(s-\frac{1}{12}\right)\left(\frac{\partial^4T}{\partial x^4}+\frac{\partial^4T}{\partial y^4}\right)+p^2\left(\frac{1}{12}-m\right)\frac{\partial^4T}{\partial t^4}+p^2\left(\frac{1}{6}-m\right)\frac{\partial^3T}{\partial t^3}\right]. \tag{8}$$

Obviously all members of the family of the scheme (7) for arbitrary m and s have an order of accuracy $O(h^2+k^2)$. For $s=1/12$, we obtain formulas of accuracy $O(h^4+k^2)$ as was expected.

A standard von Neumann stability analysis of three-level difference schemes examines the eigenvalues λ_i $(i=1, 2)$ of the amplification matrix. For formula (7) the roots of the appropriate amplification matrix are given by

$$\begin{aligned}\tau^*[1-4a(s-mp^2/\tau^*)]\,[1-4b(s-mp^2/\tau^*)]\,\lambda^2 \\ = -4p^2(1-2m)\,(a+b-8sab)\,\lambda+2\tau[1-4s(a+b)+16s^2ab]\,(\lambda-1) \\ +\tau^*[1-4a(s+mp^2/\tau^*)]\,[1-4b(s+mp^2/\tau^*)],\end{aligned} \tag{9}$$

where $a=\sin^2\beta h/2$ and $b=\sin^2\gamma h/2$. Putting

$$C_1=\tau^*[(1-4sa)\,(1-4sb)+16m^2p^4/(\tau^*)^2], \tag{10a}$$

$$C_2=4p^2(a+b-8sab), \tag{10b}$$

$$C_3=2\tau(1-4sa)\,(1-4sb), \tag{10c}$$

Eq. (9) can be transformed to

$$(C_1+C_2m)\,\lambda^2-[C_3-C_2(1-2m)]\,\lambda+C_3+C_2m-C_1=0. \tag{11}$$

Since $p>0$ and $0\leqslant a,\ b\leqslant 1$, then for $s<1/4$ all C_i $(i=1, 2, 3)$ are nonnegative.

If the discriminant of (11) is negative then $\lambda=v+\eta i$ $(v, \eta$ real), and from (11) we get

$$v^2+\eta^2=\frac{C_3-C_1+C_2m}{C_1+C_2m}. \tag{12}$$

Since the right-hand side term of (12) can be shown to be less than one, this case satisfies the condition for stability.

If the discriminant of (11) is positive, then

$$\lambda=\frac{C_3-C_2(1-4m)\pm[(C_3-2C_1)^2+C_2(1-4m)-C_2C_3]^{1/2}}{2(C_1+C_2m)}. \tag{13}$$

For $m\geqslant 1/4$ the right-hand side of (13) is always less than one and so formula (7) is always stable in the von Neumann sense if $s<1/4$ and $m\geqslant 1/4$.

A stable (independent of time-step), highly accurate ADI method developed in this note provides the basis for the convenient numerical solution of the two-dimensional hyperbolic heat conduction problems encountered in practice. The modifications required in the method to cater to variable coefficients and arbitrary regions follow a pattern similar to those outlined in Chapter 2 of [13].

References

1. C. Cattaneo, *Atti. Sem. Mat. Fis. Univ. Modena*, **3–4** (1948), 93.
2. C. Cattaneo, *C. R. Acad. Sci. Paris Ser.* **I** (1958), 431.
3. M. P. Vernotte, *C. R. Acad. Sci. Paris Ser.* **II** (1958), 2103.
4. M. P. Vernotte, *C. R. Acad. Sci. Paris* **II** (1958), 3154.
5. B. D. Coleman and W. Noll, *Arch. Ration. Mech. Anal.* **13** (1963), 167.
6. B. D. Coleman, *Arch. Ration. Mech. Anal.* **17** (1964), 1.
7. R. E. Nettleton, *Phys. Fluids* **3** (1963), 216.
8. S. Kaliski, *Acad. Pol. Sci. Ser. Sci. Tech.* **13** (1965), 253.
9. M. J. Maurer and H. A. Thompson, *J. Heat Transfer: Trans. ASME Ser.* C **95** (1973), 284.
10. E. Lorenzini and M. Spiga, *Wärme- und Stoffübertrag* **16** (1982), 113.
11. M. K. Jain, R. Ahuja and S. Bhattacharyya, *Int. J. Numer. Methods Eng.* **10** (1976), 960.
12. R. M. Beam and R. F. Warming, *J. Comput. Phys.* **22** (1976), 87.
13. A. R. Mitchell, "Computational Methods in Partial Differential Equations," Chap. 2, Wiley, London, 1969.

Received: June 13, 1984; revised: April 17, 1985

Andrzej B. Jarzębski and Jan W. Thullie

Polish Academy of Sciences,
Institute of Chemical Engineering,
44-100 Gliwice, ul. Baltycka 5, Poland